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1979 J. Phys. A: Math. Gen. 12 2187

(http://iopscience.iop.org/0305-4470/12/11/028)

J. Phys. A: Math. Gen., Vol. 12, No. 11, 1979. Printed in Great Britain

Photon splitting in the magnetised vacuum

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Received 5 December 1978

Abstract. An explicit result for the quadratic vacuum polarisation tensor in a static uniform magnetic field is derived using the Géhénian representation of the electron propagator. The formalism of relativistic-quantum plasma physics is used to calculate the probability for a photon to split into two photons in the magnetised vacuum, without approximation in the magnetic field strength or in the photon frequency or wavenumber. The exact effect of photon dispersion on photon splitting is included. It is shown that the probability for photon splitting in both the weak-field and low-frequency limits is greatest when the energy of the initial photon is divided equally between the two final photons. Errors are indicated in earlier results for the box-diagram contribution to photon splitting in the magnetised vacuum.

1. Introduction

The possible existence in pulsars of magnetic fields with strength of order the critical field strength $B_c = m^2c^3/e\hbar \approx 4{\cdot}4\times10^{13}$ gauss has generated interest in relativistic quantum processes occurring in a strong magnetic field. In particular, the splitting of a photon into two photons in the presence of a magnetic field has been discussed in various approximations by Adler *et al* (1970), Bialynicka–Birula and Bialynicki–Birula (1970) and Adler (1971). For $B \leqslant B_c$, Bialynicka–Birula and Bialynicki–Birula (1970) have shown that the splitting of a photon into more than two photons is suppressed due to the smallness of both the scattering amplitude and the available phase space. In this paper, the probability for a photon to split into two photons due to the quadratic polarisation of the magnetised vacuum is derived exactly, without approximation in the frequency ω, wavenumber $\boldsymbol{k}$ or magnetic field strength B, and without assuming that the vacuum refractive indices are approximately equal to unity (i.e. without assuming weak photon dispersion). Radiative corrections to the quadratic vacuum polarisation tensor are assumed to be negligible (a reasonable assumption for $B \leqslant B_c$).

One method of calculation of the probabilities for relativistic quantum processes in a magnetic field involves replacing the field-free electron propagator in the Feynman diagram for the corresponding field-free process by the electron propagator in the ambient magnetic field. One representation of this propagator, which is exact in the ambient field, was derived by Géhéniau (1950) and Géhéniau and Demeur (1951). Photon dispersion in the magnetised vacuum may be determined by making this replacement of the electron propagator in the 'bubble' diagram corresponding to the linear vacuum polarisation tensor (e.g. Tsai 1974, Melrose and Stoneham 1976). The

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0305-4470/79/122187+17$01.00

magnetised vacuum is found to be birefringent with the two refractive indices being functions of the field strength and of the photon frequency and wavenumber. The probability for photon–photon scattering in the magnetised vacuum may be determined by making this replacement of the electron propagator in the 'box' diagram corresponding to the cubic vacuum polarisation tensor (e.g. Ng and Tsai 1977). In this paper this replacement of the electron propagator is made in the 'triangle' diagram corresponding to the quadratic vacuum polarisation tensor. The formalism of relativistic quantum plasma physics (e.g. Melrose 1974) is then used to derive exact results for the probability for a photon to split into two photons in the presence of an ambient magnetic field. One advantage of this method of calculation is the relative ease with which the exact effect of photon dispersion on photon splitting is included.

The Feynman diagrams relevant to photon splitting are presented in § 2 and the importance of the 'non-dispersive case' (where photon dispersion is ignored) is discussed. The Géhénian representation of the electron propagator in a magnetic field is recorded in § 3 and is used in § 4 to calculate the quadratic vacuum polarisation tensor in a magnetic field. Exact probabilities for photon splitting in the magnetised vacuum are derived in § 5 using the formalism of relativistic quantum plasma physics. The non-dispersive case of photon splitting is discussed in § 6 and dispersive effects are included in § 7. In the Appendix, errors are pointed out in the results obtained by Bialynicka–Birula and Bialynicki–Birula (1970) and Adler (1971) for the box–diagram contribution to photon splitting in the magnetised vacuum.

The notation used in this paper is that of Berestetskii *et al* (1971), with the exceptions that here the electronic charge is $-e$ and Sp denotes the trace over Dirac matrices. Unrationalised Gaussian units with $\hbar = c = 1$ are used. The symbols $:=$ and $=:$ define the quantities on the left and right respectively, and $A^\mu = (A^0, \boldsymbol{A})$ relates a 4-vector to its time and space components and $\boldsymbol{A} = (A_1, A_2, A_3)$ relates the 3-vector to its Cartesian components. The quantities $g_\perp^{\mu\nu}$ and $g_\parallel^{\mu\nu}$ are defined as diagonal $(0, -1, -1, 0)$ and diagonal $(1, 0, 0, -1)$ respectively and the contractions of two 4-vectors a^μ and b^μ over the $\perp$ and $\parallel$ sub-spaces are written as $(ab)_\perp := a^\sigma b^\tau g^\perp_{\sigma\tau}$ and $(ab)_\parallel := a^\sigma b^\tau g^\parallel_{\sigma\tau}$. The $\perp$ and $\parallel$ parts of a 4-vector are defined by $a_\perp^\mu := g_\perp^{\mu\sigma} a_\sigma$ and $a_\parallel^\mu := g_\parallel^{\mu\sigma} a_\sigma$.

2. Feynman diagrams

The processes of photon dispersion and photon splitting in an ambient magnetic field are strictly inseparable processes. To lowest order in the fine structure constant α, but exactly in the ambient field, the Feynman diagrams for the splitting of a photon into two photons are given in figure 1, where the electron propagator is the Géhénian propagator and where the dispersion relations for the initial and final photons are given by the exact results for the magnetised vacuum.

When photon disperion is weak it is convenient to consider initially the non-dispersive case. Dispersive effects may then be included as small perturbations. Conservation of 4-momentum in the non-dispersive case can only be satisfied if the propagation directions of the three photons are identical. This condition, together with the requirements of Lorentz invariance, gauge invariance and charge–conjugation invariance, imposes restrictions on the scattering amplitudes for photon splitting. Bialynicka–Birula and Bialynicki–Birula (1970) and Adler (1971), for example, have shown that photon splitting in the non-dispersive case with one interaction with the ambient field (figure 2) is forbidden. Adler (1971) has also shown that the scattering amplitudes for splitting with exactly three interactions with the ambient field (figure 3) is exactly given by its low frequency limit.

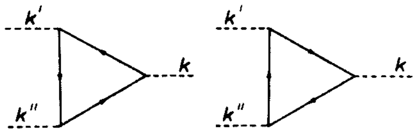

Figure 1. The Feynman diagrams for photon splitting. The electron propagator is the Géhénian propagator and the dispersion relations for the initial and final photons are given by the exact results for the magnetised vacuum.

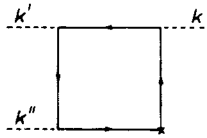

Figure 2. A box diagram for photon splitting. The × denotes an interaction with the ambient field. There are six such diagrams corresponding to permutations of the vertices.

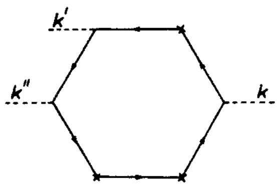

Figure 3. A hexagon diagram for photon splitting. Each × denotes an interaction with the ambient field. There are twenty such diagrams corresponding to permutations of the vertices.

3. The electron propagator

Géhéniau (1950) and Géhéniau and Demeur (1951) derived a one-dimensional integral representation of the electron propagator in a static uniform magnetic field which includes the effect of the ambient field exactly. With the 3-axis along the magnetic field $\boldsymbol{B}$ the propagator from $x^{\mu} = (t, \boldsymbol{r})$ to $x'^{\mu} = (t', \boldsymbol{r}')$ may be written as

$$G(x, x') = \phi(x, x')\Delta(x - x'), \tag{1}$$

with

$$\phi(x, x') := \exp\left\{-ie \int_x^{x'} dx_{\mu} A^{\mu}(x)\right\} \tag{2}$$

and

$$\Delta(x) := -\frac{eB}{16\pi^2} \int_0^{\infty} \frac{ds}{s} \frac{\exp(-im^2 s)}{\sin(eBs)} \exp\left\{-\frac{ieB(x^2)_{\perp}}{4\tan(eBs)} - \frac{i(x^2)_{\parallel}}{4s}\right\}$$
$$\times \left\{\left[m + \frac{1}{2s}(\gamma x)_{\parallel}\right] \exp(-i\Sigma eBs) + \frac{eB}{2\sin(eBs)}(\gamma x)_{\perp}\right\}. \tag{3}$$

The integral in (2) is along the straight line from x to x' and $A^\mu(x)$ is the 4-potential of the ambient field. Feynman's rule for avoiding the poles is understood in (3) and $\Sigma := \mathrm{i}\gamma^1\gamma^2 = \text{diagonal}\,(1, -1, 1, -1)$.

The function $\Delta(x)$ is independent of the choice of gauge of the ambient field. The gauge dependent part of the electron propagator may be rewritten as (Schwinger 1951)

$$\phi(x, x') = \exp\left\{-\mathrm{i}e\int_x^{x'} \mathrm{d}x_\mu\left[A^\mu(x) + \frac{1}{2}F^{\mu\nu}x_\nu\right] - \frac{1}{2}\mathrm{i}ex_\mu F^{\mu\nu}x'_\nu\right\}, \tag{4}$$

where $F^{\mu\nu} := \partial A^\nu/\partial x_\mu - \partial A^\mu/\partial x_\nu$ is the Maxwell tensor of the ambient field. The integral in (4) is independent of the path of integration.

4. The quadratic vacuum polarisation tensor

The unsymmetrised quadratic vacuum polarisation tensor in the presence of an ambient magnetic field is defined by

$$\alpha_1^{\mu\nu\rho}(x, x', x'') = \mathrm{i}e^3\mathrm{Sp}[\gamma^\mu G(x, x')\gamma^\nu G(x', x'')\gamma^\rho G(x'', x)] \tag{5}$$

and correspnds to the electron propagator part of one of the triangle diagrams of figure 1. The tensor is symmetrised in (16) below to include the contribution from the other diagram of figure 1. Substituting (1) with (4) for the electron propagator in (5) gives

$$\begin{aligned}\alpha_1^{\mu\nu\rho}(x, x', x'') = \mathrm{i}e^3 &\exp[\tfrac{1}{2}\mathrm{i}e(x - x')_\mu F^{\mu\nu}(x'' - x)_\nu] \\ &\times\mathrm{Sp}[\gamma^\mu\Delta(x - x')\gamma^\nu\Delta(x' - x'')\gamma^\rho\Delta(x'' - x)].\end{aligned} \tag{6}$$

This result is manifestly translationally invariant and manifestly independent of the choice of gauge of the ambient field. Choosing $x'' = 0$ in (6) and using (3) gives

$$\begin{aligned}\alpha_1^{\mu\nu\rho}(x, x') = -&\frac{\mathrm{i}e^3(eB)^3}{(16\pi^2)^3}\exp(\tfrac{1}{2}\mathrm{i}ex'_\mu F^{\mu\nu}x_\nu)\int_0^\infty\frac{\mathrm{d}s}{s}\int_0^\infty\frac{\mathrm{d}s'}{s'}\int_0^\infty\frac{\mathrm{d}s''}{s''} \\ &\times\frac{\exp[-\mathrm{i}m^2(s + s' + s'')]D_1^{\mu\nu\rho}(x, x')}{\sin(eBs)\sin(eBs')\sin(eBs'')} \\ &\times\exp\left[-\frac{\mathrm{i}}{4}\left(\frac{eB(x - x')_\perp^2}{\tan(eBs)} + \frac{eB(x')_\perp^2}{\tan(eBs')} + \frac{eB(x^2)_\perp}{\tan(eBs'')}\right.\right. \\ &\left.\left.+\frac{(x - x')_\parallel^2}{s} + \frac{(x')_\parallel^2}{s'} + \frac{(x^2)_\parallel}{s''}\right)\right],\end{aligned} \tag{7}$$

with

$$\begin{aligned}D_1^{\mu\nu\rho}(x, x') := \mathrm{Sp}\Big\{&\gamma^\mu\left[\exp(-\mathrm{i}\Sigma eBs)\left(m + \frac{(\gamma(x - x'))_\parallel}{2s}\right) + \frac{eB(\gamma(x - x'))_\perp}{2\sin(eBs)}\right] \\ &\times\gamma^\nu\left[\exp(-\mathrm{i}\Sigma eBs')\left(m + \frac{(\gamma x')_\parallel}{2s'}\right) + \frac{eB(\gamma x')_\perp}{2\sin(eBs')}\right] \\ &\times\gamma^\rho\left[\exp(-i\Sigma eBs'')\left(m - \frac{(\gamma x)_\parallel}{2s''}\right) - \frac{eB(\gamma x)_\perp}{2\sin(eBs'')}\right]\Big\}.\end{aligned} \tag{8}$$

The Fourier transform of (7) is given by

$$\alpha_1^{\mu\nu\rho}(k, k', k'') := \int \mathrm{d}^4x \int \mathrm{d}^4x' \exp(\mathrm{i}kx - \mathrm{i}k'x')\alpha_1^{\mu\nu\rho}(x, x'), \tag{9}$$

where the signs of the 4-momenta are chosen so that the identity

$$k_\mu = k'_\mu + k''_\mu \tag{10}$$

is satisfied. The relevant integrals in (9) have been evaluated by Bogoliubov and Shirkov (1959 § 14.1). One obtains

$$\begin{aligned}
&\alpha_1^{\mu\nu\rho}(k, k', k'')\\
&\quad= \frac{\mathrm{i}e^4B}{(4\pi)^2}\int_0^\infty \mathrm{d}s \int_0^\infty \mathrm{d}s' \int_0^\infty \mathrm{d}s'' \frac{\exp[-\mathrm{i}m^2(s+s'+s'')]D_1^{\mu\nu\rho}(k, k', k'')}{(s+s'+s'')\sin[eB(s+s'+s'')]}\\
&\quad\times\exp\left\{\mathrm{i}\left(\frac{\{s''(s+s')k^2 - 2s's''kk' + s'(s+s'')]k'^2\}_\parallel}{(s+s'+s'')}\right.\right.\\
&\quad\left.\left.+\frac{\{\not{S}''\sin[eB(s+s')]k^2 - 2\not{C}\not{S}'\not{S}''kk' + \not{S}'\sin[eB(s+s'')]k'^2\}_\perp}{eB\sin[eB(s+s'+s'')]}\right)\right\}\\
&\quad\times\exp\left\{\frac{2\mathrm{i}\not{S}\not{S}'\not{S}''(k\hat{k}')_\perp}{eB\sin[eB(s+s'+s'')]}\right\},
\end{aligned} \tag{11}$$

with

$$\begin{aligned}
&D_1^{\mu\nu\rho}(k, k', k'')\\
&\quad=\mathrm{Sp}\{\gamma^\mu[\exp(-\mathrm{i}\Sigma eBs)(m+(\gamma A)_\parallel)+(\gamma A)_\perp]\\
&\quad\times\gamma^\nu[\exp(-\mathrm{i}\Sigma eBs')(m+(\gamma B)_\parallel)+(\gamma B)_\perp]\\
&\quad\times\gamma^\rho[\exp(-\mathrm{i}\Sigma eBs'')(m+(\gamma C)_\parallel)+(\gamma C)_\perp]\}\\
&\quad+\mathrm{i}\,\mathrm{Sp}\left\{\gamma^\mu[\exp(-\mathrm{i}\Sigma eBs)(m+(\gamma A)_\parallel)+(\gamma A)_\perp]\right.\\
&\quad\times\gamma^\nu\left(\frac{\exp(-\mathrm{i}\Sigma eBs')\gamma_\parallel^\tau\gamma^\rho\exp(-\mathrm{i}\Sigma eBs'')\gamma_\parallel^\xi g_{\tau\xi}^\parallel}{2(s+s'+s'')}\right.\\
&\quad\left.+\frac{eB\gamma_\perp^\tau\gamma^\rho\gamma_\perp^\xi(\not{C}g_{\tau\xi}^\perp+\not{S}\hat{F}_{\tau\xi}^\perp)}{2\sin[eB(s+s'+s'')]}\right)\\
&\quad+\gamma^\nu[\exp(-\mathrm{i}\Sigma eBs')(m+(\gamma B)_\parallel)+(\gamma B)_\perp]\\
&\quad\times\gamma^\rho\left(\frac{\exp(-\mathrm{i}\Sigma eBs'')\gamma_\parallel^\xi\gamma^\mu\exp(-\mathrm{i}\Sigma eBs)\gamma_\parallel^\sigma g_{\xi\sigma}^\parallel}{2(s+s'+s'')}\right.\\
&\quad\left.+\frac{eB\gamma_\perp^\xi\gamma^\mu\gamma_\perp^\sigma(\not{C}'g_{\xi\sigma}^\perp+\not{S}'\hat{F}_{\xi\sigma}^\perp)}{2\sin[eB(s+s'+s'')]}\right)\\
&\quad+\gamma^\rho[\exp(-\mathrm{i}\Sigma eBs'')(m+(\gamma C)_\parallel)+(\gamma C)_\perp]\\
&\quad\times\gamma^\mu\left(\frac{\exp(-\mathrm{i}\Sigma eBs)\gamma_\parallel^\sigma\gamma^\mu\exp(-\mathrm{i}\Sigma eBs')\gamma_\parallel^\tau g_{\sigma\tau}^\parallel}{2(s+s'+s'')}\right.\\
&\quad\left.\left.+\frac{eB\gamma_\perp^\sigma\gamma^\nu\gamma_\perp^\tau(\not{C}''g_{\sigma\tau}^\perp+\not{S}''\hat{F}_{\sigma\tau}^\perp)}{2\sin[eB(s+s'+s'')]}\right)\right\},
\end{aligned} \tag{12}$$

where

$$\$:= \sin(eBs), \qquad \$' := \sin(eBs'), \qquad \$'' := \sin(eBs''),$$

$$\text{¢} := \cos(eBs), \qquad \text{¢}' := \cos(eBs'), \qquad \text{¢}'' := \cos(eBs''), \tag{13}$$

and

$$A_\perp^\mu := \frac{[\text{¢}'\$''k + \$'\text{¢}''k' - \$'\$''(\hat{k} - \hat{k}')]_\perp^\mu}{\sin[eB(s+s'+s'')]}, \qquad A_\parallel^\mu := \frac{[s''k + s'k']_\parallel^\mu}{(s+s'+s'')},$$

$$B_\perp^\mu := \frac{[\text{¢}\$''k - (\$\text{¢}'' + \text{¢}\$'')k' + \$\$''\hat{k}]_\perp^\mu}{\sin[eB(s+s'+s'')]}, \qquad B_\parallel^\mu := \frac{[s''k - (s+s'')k']_\parallel^\mu}{(s+s'+s'')},$$

$$C_\perp^\mu := \frac{[-(\$\text{¢}' + \text{¢}\$')k + \text{¢}\$'k' - \$\$'\hat{k}']_\perp^\mu}{\sin[eB(s+s'+s'')]}, \qquad C_\parallel^\mu := \frac{[-(s+s')k + s'k']_\parallel^\mu}{(s+s'+s'')}, \tag{14}$$

and where the notations $\hat{F}_\perp^{\mu\nu} := F^{\mu\nu}/B$ and $\hat{k}_\perp^\mu := \hat{F}_\perp^{\mu\sigma}k_\sigma$ have been introduced.

The traces in (12) may be evaluated using the usual rules for Dirac matrices (e.g. Berestetskii *et al* 1971). After an extremely tedious calcuation one obtains

$$\begin{aligned}
D_1^{\mu\nu\rho}(k, k', k'')
&= 4m^2[(A_\parallel^\mu - B_\parallel^\mu + C_\parallel^\mu)\{\cos[eB(s+s'+s'')]g^{\nu\rho} \\
&\quad +2\$' \sin[eB(s''+s)]g_\perp^{\nu\rho} + \sin[eB(s''+s-s')]\hat{F}_\perp^{\nu\rho}\} \\
&\quad +(A_\parallel^\nu + B_\parallel^\nu - C_\parallel^\nu)\{\cos[eB(s+s'+s'')]g^{\rho\mu} \\
&\quad +2\$'' \sin[eB(s+s')]g_\perp^{\rho\mu} + \sin[eB(s+s'-s'')]\hat{F}_\perp^{\rho\mu}\} \\
&\quad +(-A_\parallel^\rho + B_\parallel^\rho + C_\parallel^\rho)\{\cos[eB(s+s'+s'')]g^{\mu\nu} \\
&\quad +2\$ \sin[eB(s'+s'')]g_\perp^{\mu\nu} + \sin[eB(s'+s''-s)]\hat{F}_\perp^{\mu\nu}\} \\
&\quad +A_\perp^\mu\{\cos[eB(s'+s'')]g^{\nu\rho} + 2\$'\$''g_\perp^{\nu\rho} - \sin[eB(s'-s'')]\hat{F}_\perp^{\nu\rho}\} \\
&\quad +A_\perp^\nu\{\cos[eB(s'+s'')]g^{\rho\mu} + 2\$'\$''g_\perp^{\rho\mu} + \sin[eB(s'-s'')]\hat{F}_\perp^{\rho\mu}\} \\
&\quad -A_\perp^\rho \cos[eB(s'-s'')]g^{\mu\nu} + (\hat{A}_\perp^\mu g_\parallel^{\nu\rho} - \hat{A}_\perp^\nu g_\parallel^{\rho\mu}) \sin[eB(s'+s'')] \\
&\quad -\hat{A}_\perp^\rho \sin[eB(s'-s'')]g^{\mu\nu} \\
&\quad +B_\perp^\nu\{\cos[eB(s''+s)]g^{\rho\mu} + 2\$''\$g_\perp^{\rho\mu} - \sin[eB(s''-s)]\hat{F}_\perp^{\rho\mu}\} \\
&\quad +B_\perp^\rho\{\cos[eB(s''+s)]g^{\mu\nu} + 2\$''\$g_\perp^{\mu\nu} + \sin[eB(s''-s)]\hat{F}_\perp^{\mu\nu}\} \\
&\quad -B_\perp^\mu \cos[eB(s''-s)]g^{\nu\rho} + (\hat{B}_\perp^\nu g_\parallel^{\rho\mu} - \hat{B}_\perp^\rho g_\parallel^{\mu\nu}) \sin[eB(s''+s)] \\
&\quad -\hat{B}_\perp^\mu \sin[eB(s''-s)]g^{\nu\rho} \\
&\quad +C_\perp^\rho\{\cos[eB(s+s')]g^{\mu\nu} + 2\$\$'g_\perp^{\mu\nu} - \sin[eB(s-s')]\hat{F}_\perp^{\mu\nu}\} \\
&\quad +C_\perp^\mu\{\cos[eB(s+s')]g^{\nu\rho} + 2\$\$'g_\perp^{\nu\rho} + \sin[eB(s-s')]\hat{F}_\perp^{\nu\rho}\} \\
&\quad -C_\perp^\nu \cos[eB(s-s')]g^{\rho\mu} + (\hat{C}_\perp^\rho g_\parallel^{\mu\nu} - \hat{C}_\perp^\mu g_\parallel^{\nu\rho}) \sin[eB(s+s')] \\
&\quad -\hat{C}_\perp^\nu \sin[eB(s-s')]g^{\rho\mu}] \\
&\quad +4[2\cos[eB(s+s'+s'')](A_\parallel^\mu B_\parallel^\nu C_\parallel^\rho + C_\parallel^\mu A_\parallel^\nu B_\parallel^\rho) \\
&\quad -((BC)_\parallel A_\parallel^\mu - (CA)_\parallel B_\parallel^\mu + (AB)_\parallel C_\parallel^\mu)\{\cos[eB(s+s'+s'')]g^{\nu\rho}
\end{aligned}$$

$$
\begin{aligned}
&+2\$' \sin[eB(s''+s)]g_\perp^{\nu\rho}+\sin[eB(s''+s-s')]\hat{F}_\perp^{\nu\rho}\} \\
&-((BC)_\parallel A_\parallel^\nu+(CA)_\parallel B_\parallel^\nu-(AB)_\parallel C_\parallel^\nu)\{\cos[eB(s+s'+s'')]g^{\rho\mu} \\
&+2\$'' \sin[eB(s+s')]g_\perp^{\rho\mu}+\sin[eB(s+s'-s'')]\hat{F}_\perp^{\nu\mu}\} \\
&-(-(BC)_\parallel A_\parallel^\rho+(CA)_\parallel B_\parallel^\rho+(AB)_\parallel C_\parallel^\rho)\{\cos[eB(s+s'+s'')]g^{\mu\nu} \\
&+2\$ \sin[eB(s'+s'')]g_\perp^{\mu\nu}+\sin[eB(s'+s''-s)]\hat{F}_\perp^{\mu\nu}\} \\
&+(\cos[eB(s'+s'')]A_\perp^\mu+\sin[eB(s'+s'')]\hat{A}_\perp^\mu)(B_\parallel^\nu C_\parallel^\rho+C_\parallel^\nu B_\parallel^\rho-(BC)_\parallel g_\parallel^{\nu\rho}) \\
&+(\cos[eB(s'+s'')]A_\perp^\nu-\sin[eB(s'+s'')]\hat{A}_\perp^\nu)(B_\parallel^\rho C_\parallel^\mu+C_\parallel^\rho B_\parallel^\mu-(BC)_\parallel g_\parallel^{\rho\mu}) \\
&+(\cos[eB(s'-s'')]A_\perp^\rho+\sin[eB(s'-s'')]\hat{A}_\perp^\rho)(-B_\parallel^\mu C_\parallel^\nu+C_\parallel^\mu B_\parallel^\nu+(BC)_\parallel g_\parallel^{\mu\nu}) \\
&-(BC)_\parallel(\cos[eB(s'-s'')](A_\perp^\mu g_\perp^{\nu\rho}+A_\perp^\nu g_\perp^{\rho\mu}-A_\perp^\rho g_\perp^{\mu\nu}) \\
&-\sin[eB(s'-s'')](\hat{A}_\perp^\mu g_\perp^{\nu\rho}+\hat{A}_\perp^\nu g_\perp^{\rho\mu}-\hat{A}_\perp^\rho g_\perp^{\mu\nu})) \\
&+(\cos[eB(s''+s)]B_\perp^\nu+\sin[eB(s''+s)]\hat{B}_\perp^\nu)(C_\parallel^\rho A_\parallel^\mu+A_\parallel^\rho C_\parallel^\mu-(CA)_\parallel g_\parallel^{\rho\mu}) \\
&+(\cos[eB(s''+s)]B_\perp^\rho-\sin[eB(s''+s)]\hat{B}_\perp^\rho)(C_\parallel^\mu A_\parallel^\nu+A_\parallel^\mu C_\parallel^\nu-(CA)_\parallel g_\parallel^{\mu\nu}) \\
&+(\cos[eB(s''-s)]B_\perp^\mu+\sin[eB(s''-s)]\hat{B}_\perp^\mu)(-C_\parallel^\nu A_\parallel^\rho+A_\parallel^\nu C_\parallel^\rho+(CA)_\parallel g_\parallel^{\nu\rho}) \\
&-(CA)_\parallel(\cos[eB(s''-s)](B_\perp^\nu g_\perp^{\rho\mu}+B_\perp^\rho g_\perp^{\mu\nu}-B_\perp^\mu g_\perp^{\nu\rho}) \\
&-\sin[eB(s''-s)](\hat{B}_\perp^\nu g_\perp^{\rho\mu}+\hat{B}_\perp^\rho g_\perp^{\mu\nu}-\hat{B}_\perp^\mu g_\perp^{\nu\rho})) \\
&+(\cos[eB(s+s')]C_\perp^\rho+\sin[eB(s+s')]\hat{C}_\perp^\rho)(A_\parallel^\mu B_\parallel^\nu+B_\parallel^\mu A_\parallel^\nu-(AB)_\parallel g_\parallel^{\mu\nu}) \\
&+(\cos[eB(s+s')]C_\perp^\mu-\sin[eB(s+s')]\hat{C}_\perp^\mu)(A_\parallel^\nu B_\parallel^\rho+B_\parallel^\nu A_\parallel^\rho-(AB)_\parallel g_\parallel^{\nu\rho}) \\
&+(\cos[eB(s-s')]C_\perp^\nu+\sin[eB(s-s')]\hat{C}_\perp^\nu)(-A_\parallel^\rho B_\parallel^\mu+B_\parallel^\rho A_\parallel^\mu+(AB)_\parallel g_\parallel^{\rho\mu}) \\
&-(AB)_\parallel(\cos[eB(s-s')](C_\perp^\rho g_\perp^{\mu\nu}+C_\perp^\mu g_\perp^{\nu\rho}-C_\perp^\nu g_\perp^{\rho\mu}) \\
&-\sin[eB(s-s')](\hat{C}_\perp^\rho g_\perp^{\mu\nu}+\hat{C}_\perp^\mu g_\perp^{\nu\rho}-\hat{C}_\perp^\nu g_\perp^{\rho\mu})) \\
&+A_\parallel^\mu\{¢(B_\perp^\nu C_\perp^\rho+C_\perp^\nu B_\perp^\rho-(BC)_\perp g^{\nu\rho})+\$(\hat{C}_\perp^\nu B_\perp^\rho+C_\perp^\nu\hat{B}_\perp^\rho-(B\hat{C})_\perp g_\parallel^{\nu\rho})\} \\
&+A_\parallel^\nu\{¢(B_\perp^\rho C_\perp^\mu+C_\perp^\rho B_\perp^\mu-(BC)_\perp g^{\rho\mu})-\$(\hat{C}_\perp^\rho B_\perp^\mu+C_\perp^\rho\hat{B}_\perp^\mu+(B\hat{C})_\perp g_\parallel^{\rho\mu})\} \\
&+A_\parallel^\rho\{¢(-B_\perp^\mu C_\perp^\nu+C_\perp^\mu B_\perp^\nu+(BC)_\perp g^{\mu\nu})-\$(\hat{C}_\perp^\mu B_\perp^\nu-C_\perp^\mu\hat{B}_\perp^\nu-(B\hat{C})_\perp g_\parallel^{\mu\nu})\} \\
&+B_\parallel^\nu\{¢'(C_\perp^\rho A_\perp^\mu+A_\perp^\rho C_\perp^\mu-(CA)_\perp g^{\rho\mu})+\$'(\hat{A}_\perp^\rho C_\perp^\mu+A_\perp^\rho\hat{C}_\perp^\mu-(C\hat{A})_\perp g_\parallel^{\rho\mu})\} \\
&+B_\parallel^\rho\{¢'(C_\perp^\mu A_\perp^\nu+A_\perp^\mu C_\perp^\nu-(CA)_\perp g^{\mu\nu})-\$'(\hat{A}_\perp^\mu C_\perp^\nu+A_\perp^\mu\hat{C}_\perp^\nu+(C\hat{A})_\perp g_\parallel^{\mu\nu})\} \\
&+B_\parallel^\mu\{¢'(-C_\perp^\nu A_\perp^\rho+A_\perp^\nu C_\perp^\rho+(CA)_\perp g^{\nu\rho}) \\
&\qquad\qquad\qquad\qquad\qquad -\$'(\hat{A}_\perp^\nu C_\perp^\rho-A_\perp^\nu\hat{C}_\perp^\rho-(C\hat{A})_\perp g_\parallel^{\nu\rho})\} \\
&+C_\parallel^\rho\{¢''(A_\perp^\mu B_\perp^\nu+B_\perp^\mu A_\perp^\nu-(AB)_\perp g^{\mu\nu})+\$''(\hat{B}_\perp^\mu A_\perp^\nu+B_\perp^\mu\hat{A}_\perp^\nu-(A\hat{B})_\perp g_\parallel^{\mu\nu})\} \\
&+C_\parallel^\mu\{¢''(A_\perp^\nu B_\perp^\rho+B_\perp^\nu A_\perp^\rho-(AB)_\perp g^{\nu\rho}) \\
&-\$''(\hat{B}_\perp^\nu A_\perp^\rho+B_\perp^\nu\hat{A}_\perp^\rho+(A\hat{B})_\perp g_\parallel^{\nu\rho})\} \\
&+C_\parallel^\nu\{¢''(-A_\perp^\rho B_\perp^\mu+B_\perp^\rho A_\perp^\mu+(AB)_\perp g^{\rho\mu}) \\
&\qquad\qquad\qquad\qquad\qquad -\$''(\hat{B}_\perp^\rho A_\perp^\mu-B_\perp^\rho\hat{A}_\perp^\mu-(A\hat{B})_\perp g_\parallel^{\rho\mu})\} \\
&+2(A_\perp^\mu B_\perp^\nu C_\perp^\rho+C_\perp^\mu A_\perp^\nu B_\perp^\rho)-((BC_\perp A_\perp^\mu-(CA)_\perp B_\perp^\mu+(AB)_\perp C_\perp^\mu)g^{\nu\rho}
\end{aligned}
$$

$$
\begin{aligned}
&-((BC)_\perp A^\nu_\perp+(CA)_\perp B^\nu_\perp-(AB)_\perp C^\nu_\perp)g^{\rho\mu}\\
&-(-(BC)_\perp A^\rho_\perp+(CA)_\perp B^\rho_\perp+(AB)_\perp C^\rho_\perp)g^{\mu\nu}]\\
&+\frac{4\mathrm{i}}{(s+s'+s'')}[\cos[eB(s'-s'')](A^\rho_\perp g^{\mu\nu}-A^\mu_\perp g^{\nu\rho}_\perp-A^\nu_\perp g^{\rho\mu}_\perp)\\
&+\sin[eB(s'-s'')](\hat{A}^\rho_\perp(g^{\mu\nu}_\parallel-g^{\mu\nu}_\perp)+\hat{A}^\mu_\perp g^{\nu\rho}_\perp+\hat{A}^\nu_\perp g^{\rho\nu}_\perp)\\
&+\cos[eB(s''-s)](B^\mu_\perp g^{\nu\rho}-B^\nu_\perp g^{\rho\mu}_\perp-B^\rho_\perp g^{\mu\nu}_\perp)\\
&+\sin[eB(s''-s)](\hat{B}^\mu_\perp(g^{\nu\rho}_\parallel-g^{\nu\rho}_\perp)+\hat{B}^\nu_\perp g^{\rho\mu}_\perp+\hat{B}^\rho_\perp g^{\mu\nu}_\perp)\\
&+\cos[eB(s-s')](C^\nu_\perp g^{\rho\mu}-C^\rho_\perp g^{\mu\nu}_\perp-C^\mu_\perp g^{\nu\rho}_\perp)\\
&+\sin[eB(s-s')](\hat{C}^\nu_\perp(g^{\rho\mu}_\parallel-g^{\rho\mu}_\perp)+\hat{C}^\rho_\perp g^{\mu\nu}_\perp+\hat{C}^\mu_\perp g^{\nu\rho}_\perp)\\
&-(B^\rho_\parallel+C^\rho_\parallel)\{\cos[eB(s'+s''-s)]g^{\mu\nu}_\perp+\sin[eB(s'+s''-s)]\hat{F}^{\mu\nu}_\perp\}\\
&-(C^\mu_\parallel+A^\mu_\parallel)\{\cos[eB(s''+s-s')]g^{\nu\rho}_\perp+\sin[eB(s''+s-s')]\hat{F}^{\nu\rho}_\perp\}\\
&-(A^\nu_\parallel+B^\nu_\parallel)\{\cos[eB(s+s'-s'')]g^{\rho\mu}_\perp+\sin[eB(s+s'-s'')]\hat{F}^{\rho\mu}_\perp\}]\\
&+\frac{4\mathrm{i}eB}{\sin[eB(s+s'+s'')]}[A^\rho_\parallel(g^{\mu\nu}_\parallel+\cos(2eBs)g^{\mu\nu}_\perp-\sin(2eBs)\hat{F}^{\mu\nu}_\perp)\\
&+B^\mu_\parallel(g^{\nu\rho}_\parallel+\cos(2eBs')g^{\nu\rho}_\perp-\sin(2eBs')\hat{F}^{\nu\rho}_\perp)\\
&+C^\nu_\parallel(g^{\rho\mu}_\parallel+\cos(2eBs'')g^{\rho\mu}_\perp-\sin(2eBs'')\hat{F}^{\rho\mu}_\perp)\\
&-A^\mu_\parallel g^{\nu\rho}_\parallel-A^\nu_\parallel g^{\rho\mu}_\parallel-B^\nu_\parallel g^{\rho\mu}_\parallel-B^\rho_\parallel g^{\mu\nu}_\parallel-C^\rho_\parallel g^{\mu\nu}_\parallel-C^\mu_\parallel g^{\nu\rho}_\parallel\\
&-\text{¢}(A^\mu_\perp g^{\nu\rho}_\parallel+A^\nu_\perp g^{\rho\mu}_\parallel)-\text{¢}'(B^\nu_\perp g^{\rho\mu}_\parallel+B^\rho_\perp g^{\mu\nu}_\parallel)-\text{¢}''(C^\rho_\perp g^{\mu\nu}_\parallel+C^\mu_\perp g^{\nu\rho}_\parallel)\\
&+\$(\hat{A}^\mu_\perp g^{\nu\rho}-\hat{A}^\nu_\perp g^{\rho\mu}-A^\rho_\perp\hat{F}^{\mu\nu}_\perp)+\$'(\hat{B}^\nu_\perp g^{\rho\mu}-\hat{B}^\rho_\perp g^{\mu\nu}-B^\mu_\perp\hat{F}^{\nu\rho}_\perp)\\
&+\$''(\hat{C}^\rho_\perp g^{\mu\nu}-\hat{C}^\mu_\perp g^{\nu\rho}-C^\nu_\perp\hat{F}^{\rho\mu}_\perp)].
\end{aligned}
\qquad (15)
$$

The symmetrised quadratic vacuum polarisation tensor is defined by

$$\alpha^{\mu\nu\rho}(k,k',k'')=\tfrac{1}{2}[\alpha^{\mu\nu\rho}_1(k,k',k'')+\alpha^{\mu\rho\nu}_1(k,k'',k')]. \qquad (16)$$

For the magnetised vacuum, (16) with (11) may be rewritten as

$$
\begin{aligned}
&\alpha^{\mu\nu\rho}(k,k',k'')\\
&=\frac{\mathrm{i}e^4B}{2(4\pi)^2}\int_0^\infty \mathrm{d}s\int_0^\infty \mathrm{d}s'\int_0^\infty \mathrm{d}s''\,\frac{\exp[-\mathrm{i}m^2(s+s'+s'')]}{(s+s'+s'')\sin[eB(s+s'+s'')]}\\
&\times\exp\bigg\{\mathrm{i}\bigg(\frac{[s''sk^2+ss'k'^2+s's''k''^2]_\parallel}{(s+s'+s'')}\\
&+\frac{[\$''\$\text{¢}'k^2+\$\$'\text{¢}''k'^2+\$'\$''\text{¢}k''^2]_\perp}{eB\sin[eB(s+s'+s'')]}\\
&\times\bigg\{D^{\mu\nu\rho}_1(k,k',k'';s,s',s'')\exp\bigg(\frac{2\mathrm{i}\$\$'\$''(k''\hat{k}')_\perp}{eB\sin[eB(s+s'+s'')]}\bigg)\\
&+D^{\mu\rho\nu}_1(k,k'',k';s'',s',s)\exp\bigg(\frac{-2\mathrm{i}\$\$'\$''(k''\hat{k}')_\perp}{eB\sin[eB(s+s'+s'')]}\bigg)\bigg\},
\end{aligned}
\qquad (17)
$$

where the dependence of $D_1^{\mu\nu\rho}$ on the integration variables has been indicated explicitly and a convenient relabelling of the integration variables s and s'' has been made.

The symmetrised tensor (17) satisfies

$$\alpha^{\mu\nu\rho}(0, 0, 0) = 0 \tag{18}$$

and, in the absence of a magnetic field, (17) vanishes identically, as required by Furry's (1937) theorem. The symmetrised tensor is therefore both finite and gauge invariant and does not need to be renormalised.

An alternative form of $\alpha^{\mu\nu\rho}(k, k', k'')$ may be obtained from (17) by the transformation $s \to -\mathrm{i}s$, $s' \to -\mathrm{i}s'$, $s'' \to -\mathrm{i}s''$:

$$\begin{aligned}\alpha^{\mu\nu\rho}&(k, k', k'')\\ &= \frac{e^4 B}{2(4\pi)^2} \int_0^\infty \mathrm{d}s \int_0^\infty \mathrm{d}s' \int_0^\infty \mathrm{d}s'' \frac{\exp[-m^2(s+s'+s'')]}{(s+s'+s'')\sinh[eB(s+s'+s'')]}\\ &\quad\times \exp\left\{\frac{[s''s\,k^2 + ss'\,k'^2 + s's''\,k''^2]_\parallel}{(s+s'+s'')} + \frac{[\$''\$\text{¢}'k^2 + \$\$'\text{¢}''k'^2 + \$'\$''\text{¢}k''^2]_\perp}{eB\sinh[eB(s+s'+s'')]}\right\}\\ &\quad\times\left\{D_1^{\mu\nu\rho}(k, k', k''; -\mathrm{i}s, -\mathrm{i}s', -\mathrm{i}s'')\exp\left(\frac{-2\mathrm{i}\$\$'\$''(k''\hat{k}')_\perp}{eB\sinh[eB(s+s'+s'')]}\right)\right.\\ &\quad\left.\times D_1^{\mu\rho\nu}(k, k'', k'; -\mathrm{i}s'', -\mathrm{i}s', -\mathrm{i}s)\exp\left(\frac{2\mathrm{i}\$\$'\$''(k''\hat{k}')_\perp}{eB\sinh[eB(s+s'+s'')]}\right)\right\}\end{aligned} \tag{19}$$

where

$$\$:= \sinh(eBs), \qquad \text{¢} := \cosh(eBs), \tag{20}$$

and so on. The substitution of the upper integration limits by infinity in (19) is achieved by rotating through $\pi/2$ in each integration plane and this is possible when no poles are encountered. The quadratic vacuum polarisation tensor has poles when resonant processes contribute to the quadratic polarisation of the vacuum. A sufficient condition for the absence of poles is that photon-induced pair production be impossible.

The quadratic vacuum polarisation tensor must satisfy certain symmetry properties (e.g. Melrose 1972). The relation

$$\alpha^{\mu\nu\rho}(k, k', k'') = \alpha^{\mu\rho\nu}(k, k'', k') \tag{21}$$

is satisfied by construction. The crossing symmetry

$$\alpha^{\mu\nu\rho}(k, k', k'') = \alpha^{\nu\mu\rho}(-k', -k, k'') \tag{22}$$

for the symmetrised tensor follows from the identity

$$D_1^{\nu\mu\rho}(-k', -k, k''; s, s'', s') = D_1^{\mu\rho\nu}(k, k'', k'; s'', s', s). \tag{23}$$

The time-reversal invariance relation

$$\alpha_{ijl}(\omega, \boldsymbol{k}; \omega', \boldsymbol{k}'; \omega'', \boldsymbol{k}''; \boldsymbol{B}) = -\alpha_{ijl}(-\omega, \boldsymbol{k}; -\omega', \boldsymbol{k}'; -\omega'', \boldsymbol{k}''; -\boldsymbol{B}) \tag{24}$$

for the 3-tensor α_{ijl} which is equal to the $\mu = i$, $\nu = j$, $\rho = l$ component of $\alpha^{\mu\nu\rho}$ follows from the identity

$$D_1^{\mu\nu\rho}(k, k', k''; s, s', s''; -F) = -D_1^{\mu\rho\nu}(k, k'', k'; s'', s', s; F) \tag{25}$$

ω, ω' and ω''. The identity (25) may also be used to show that (17) satisfies the symmetry relation

$$\alpha^{\mu\nu\rho}(k, k', k''; F) = -\alpha^{\mu\nu\rho}(k, k', k''; -F), \tag{26}$$

which is a necessary consequence of the charge–conjugation invariance of quantum electrodynamics (Furry 1937).

5. The photon splitting probability

The probability for the splitting of a photon into two photons may be obtained from the quadratic vacuum polarisation tensor by a procedure familiar in plasma physics (e.g. Melrose 1974). The principal modes of propagation for a photon in the magnetised vacuum have electric polarisation vectors either perpendicular ($\perp$) or parallel ($\|$) to the plane formed by $\boldsymbol{k}$ and $\boldsymbol{B}$. The scattering amplitude for a photon in the mode σ to split into photons in the modes σ' and σ'' is

$$M(\sigma \to \sigma' + \sigma'') = 2(4\pi)^{3/2} e_\mu^\sigma(\boldsymbol{k}) e_\nu^{\sigma'*}(\boldsymbol{k}') e_\rho^{\sigma''*}(\boldsymbol{k}'') \alpha^{\mu\nu\rho}(k, k', k''), \tag{27}$$

where $e_\mu^\sigma(\boldsymbol{k})$ is the polarisation 4-vector of a photon in the mode σ, $*$ denotes complex conjugation and the factor of 2 is due to the indistinguishability of the two final photons. The probability per unit time for the decay process $\sigma \to \sigma' + \sigma''$ for final photons in the ranges $d^3\boldsymbol{k}'/(2\pi)^3$ and $d^3\boldsymbol{k}''/(2\pi)^3$ is

$$w_\sigma^{\sigma'\sigma''}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'') = \frac{R_E^\sigma(\boldsymbol{k}) R_E^{\sigma'}(\boldsymbol{k}') R_E^{\sigma''}(\boldsymbol{k}'')}{|\omega^\sigma(\boldsymbol{k})\omega^{\sigma'}(\boldsymbol{k}')\omega^{\sigma''}(\boldsymbol{k}'')|} |M(\sigma \to \sigma' + \sigma'')|^2 (2\pi)^4 \delta^{(4)}(k - k' - k''), \tag{28}$$

where $R_E^\sigma(\boldsymbol{k})$ is the ratio of electric to total energy for photons in the mode σ. The absorption coefficients due to photon splitting may be obtained from (28) by integrating over the momenta of the final photons. This gives

$$\kappa_\sigma^{\sigma'\sigma''}(\boldsymbol{k}) = \frac{1}{2} \int \frac{d^3\boldsymbol{k}'}{(2\pi)^3} \int \frac{d^3\boldsymbol{k}''}{(2\pi)^3} w_\sigma^{\sigma'\sigma''}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}''), \tag{29}$$

where the factor of $\frac{1}{2}$ is due to the indistinguishability of the two final photons.

The probability for photon splitting in the magnetised vacuum to lowest order in the radiation field but exactly in the ambient field is given by (28) with the quadratic vacuum polarisation tensor identified as the symmetrised tensor (17) and with the wave properties of the magnetised vacuum determined from the linear vacuum polarisation tensor (e.g. Melrose and Stoneham 1976). This result is valid for all magnetic field strengths and for all photon frequencies and wavenumbers provided radiative corrections are negligible. It is a generalisation of results obtained earlier by Adler *et al* (1970), Bialynicka–Birula and Bialynicki–Birula (1970) and Adler (1971).

The lowest-order term in an expansion of the scattering amplitude for photon splitting in powers of the ambient field strength is proportional to B. This term corresponds to the box diagram of figure 2 with photon dispersion included. It is identically zero in the non-dispersive case (see § 6). Explicit results for this term are given in the Appendix in the weak-field weak-dispersion limit. The next order term in an expansion of the scattering amplitude is proportional to B^3. This term corresponds to the hexagon diagram of figure 3 with photon dispersion included and is the dominant term in the weak-field weak-dispersion limit. Odd powers of the ambient field strength do not contribute to the scattering amplitude due to the charge–conjunction invariance of quantum electrodynamics.

In the strong-field limit $B \gg B_c$, the scatttering amplitude for photon splitting is of order $\exp(-B/B_c)$. Photon splitting into two photons is suppressed in this limit. However, the splitting of a photon into more than two photons may be significant.

6. The non-dispersive case

In the non-dispersive case, the probability (28) for photon splitting reduces to

$$w_{\sigma}^{\sigma'\sigma''}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'') = \frac{|M(\sigma \to \sigma' + \sigma'')|^2}{8\omega\omega'\omega''} (2\pi)^4 \delta^{(4)}(k - k' - k'') \tag{30}$$

and the absorption coefficients due to photon splitting reduce to

$$\kappa_{\sigma}^{\sigma'\sigma''}(\omega) = \frac{1}{32\pi\omega^2} \int_0^{\omega} \mathrm{d}\omega' \int_0^{\omega} \mathrm{d}\omega''\, \delta(\omega - \omega' - \omega'') |M(\sigma \to \sigma' + \sigma'')|^2. \tag{31}$$

In this case, the only Lorentz invariants on which the scattering amplitudes for photon splitting can depend are B^2, $\omega^2 \sin^2 \theta$, $\omega'^2 \sin^2 \theta$ and $\omega''^2 \sin^2 \theta$, where θ is the angle between the photon propagation direction and $\boldsymbol{B}$. Hence, to calculate the scattering amplitudes for arbitrary θ one may perform the calcuation for $\theta = \pi/2$ and then replace ω by $\omega \sin \theta$, ω' by $\omega' \sin \theta$ and ω'' by $\omega'' \sin \theta$. The scattering amplitudes derived from (17) for $\theta = \pi/2$ in the non-dispersive case reduce to

$$\begin{aligned} M(\sigma \to \sigma' + \sigma'') &\\ = -\frac{\mathrm{i}e^4 B}{\pi^{1/2}} \int_0^{\infty} \mathrm{d}s \int_0^{\infty} \mathrm{d}s' \int_0^{\infty} \mathrm{d}s'' \frac{\exp(-\mathrm{i}m^2 S) D(\sigma \to \sigma' + \sigma'')}{S^3 \sin^4(eBS)} &\\ \times \exp\left\{\mathrm{i}\left(\frac{[s''s\omega^2 + ss'\omega'^2 + s's''\omega''^2]}{S}\right.\right. &\\ \left.\left. - \frac{[\$''\$¢'\omega^2 + \$\$'¢''\omega'^2 + \$'\$''¢\omega''^2]}{eB \sin(eBS)}\right)\right\}, \end{aligned} \tag{32}$$

with

$$\begin{aligned} &D[(\perp) \to (\|)' + (\|)''] = \bar{D}_1(\omega, \omega', \omega''; s, s', s''), \\ &D[(\|) \to (\perp)' + (\|)''] = \bar{D}_1(-\omega', \omega'', -\omega; s', s'', s), \\ &D[(\|) \to (\|)' + (\perp)''] = \bar{D}_1(-\omega'', -\omega, \omega'; s'', s, s'), \\ &D[(\perp) \to (\perp)' + (\perp)''] = \bar{D}_2(\omega, \omega', \omega''; s, s', s''), \\ &D(\sigma \to \sigma' + \sigma'') = 0 \text{ otherwise}; \end{aligned} \tag{33}$$

where

$$S := s + s' + s'', \tag{34}$$

$$
\begin{aligned}
&\bar{D}_1(\omega, \omega', \omega''; s, s', s'')\\
&\quad = 4m^2S^2 \sin^2(eBS)[(\$'\text{¢}'' + \text{¢}'\$'')^2\omega' + (\$'\text{¢} + \text{¢}'\$)^2\omega'']\\
&\qquad + 4iS \sin^2(eBS)(\$^2\omega' + \$''^2\omega'')\\
&\qquad + 8ieBS^2 \sin(eBS)[\$(\$'\text{¢}'' + \text{¢}'\$'')\omega' + \$''(\$'\text{¢} + \text{¢}'\$)\omega'']\\
&\qquad + 4 \sin^2(eBS)\{-(\$'\text{¢}'' + \text{¢}'\$'')^2(s\omega' - s''\omega'')[s\omega' + (s+s')\omega'']\omega'\\
&\qquad + (\$'\text{¢} + \text{¢}'\$)^2(s\omega' - s''\omega'')[(s'+s'')\omega' + s''\omega'']\omega''\\
&\qquad - \omega'\omega''S[\$^2[(s'+s'')\omega' + s''\omega''] + \$''^2[s\omega' + (s+s')\omega'']]\}\\
&\qquad + 4S^2\{\$^2(\$'\text{¢}'' + \text{¢}'\$'')^2\omega'^3 + \$^2\$''[\$'' + 2\text{¢}'(\$'\text{¢}'' + \text{¢}'\$'')]\omega'^2\omega''\\
&\qquad + \$\$''^2[\$ + 2\text{¢}'(\$'\text{¢} + \text{¢}'\$)]\omega'\omega''^2 + \$''^2(\$'\text{¢} + \text{¢}'\$)^2\omega''^3\},\\
&\bar{D}_2(\omega, \omega', \omega''; s, s', s'')\\
&\quad = -4 \sin^2(eBS)\{m^2S^2 - iS - (s\omega' - s''\omega'')[s\omega' + (s+s')\omega'']\}\\
&\qquad \times\{(\$'^2\text{¢}''^2 - \text{¢}'^2\$''^2)\omega' + [2\$'\text{¢}'\$''\text{¢}'' + \$''^2(\$'^2 - \text{¢}'^2)]\omega''\}\\
&\qquad + 4 \sin^2(eBS)\{m^2S^2 - iS + [s\omega' + (s+s')\omega''][(s'+s'')\omega' + s''\omega'']\}\\
&\qquad \times\{[2\$\text{¢}\$''\text{¢}'' + \$^2(\$''^2 - \text{¢}''^2)]\omega' + [2\$\text{¢}\$''\text{¢}'' + \$''^2(\$^2 - \text{¢}^2)]\omega''\}\\
&\qquad - 4 \sin^2(eBS)\{m^2S^2 - iS + [(s'+s'')\omega' + s''\omega''](s\omega' - s''\omega'')\}\\
&\qquad \times\{[2\$\text{¢}\$'\text{¢}' + \$^2(\$'^2 - \text{¢}'^2)]\omega' + (\$'^2\text{¢}^2 - \text{¢}'^2\$^2)\omega''\}\\
&\qquad + 4S^2\{\$^2(\$'^2\text{¢}''^2 - \text{¢}'^2\$''^2)\omega'^3 + \$''^2(\$'^2\text{¢}^2 - \text{¢}'^2\$^2)\omega''^3\\
&\qquad - [\$^2\$''^2 + 2\$\$''(\$'\text{¢} + \text{¢}'\$)(\$'\text{¢}'' + \text{¢}'\$'')\omega\omega'\omega''\}. \qquad (35)
\end{aligned}
$$

In the weak-field limit the leading contribution to the scattering amplitudes is of order B^3. The linear dependence of the weak-field scattering amplitude on the magnetic field strength found by Skobov (1959), Minguzzi (1961), Sannikov (1967) and Gal'tsov and Skobelev (1971) is spurious. These authors incorrectly took account of the gauge dependent term $\phi(x', x'')$ in the electron propagator (1).

To lowest order in the magnetic field strength the scattering amplitudes are proportional to $\omega\omega'\omega''$ and higher-order terms in ω, ω' and ω'' are absent. This confirms the statement by Adler (1971) that the scattering amplitude for photon splitting with exactly three interactions with the ambient field is exactly given by its low-frequency limit. The probability for photon splitting in the weak-field limit is greatest when the energy of the initial photon is divided equally between the two final photons (i.e. when $\omega' = \omega'' = \frac{1}{2}\omega$).

An alternative form of the scattering amplitudes may be obtained by changing integration variables to

$$
s := i(s + s' + s''), \qquad t := i(s + s'), \qquad u := is. \tag{36}
$$

This gives

$$
\begin{aligned}
&M(\sigma \to \sigma' + \sigma'')\\
&\quad = \frac{ie^4B}{\pi^{1/2}} \int_0^\infty ds \int_0^s dt \int_0^t du \frac{\exp(-m^2s)D^\dagger(\sigma \to \sigma' + \sigma'')}{s^3 \sinh^4(eBs)}
\end{aligned}
$$

$$\times\exp\left\{\left(\frac{u(s-t)}{s}-\frac{\sinh(eBu)\sinh[eB(s-t)]\cosh[eB(t-u)]}{eB\sinh(eBs)}\right)\omega^2\right.$$

$$+\left(\frac{u(t-u)}{s}-\frac{\sinh(eBu)\sinh[eB(t-u)]\cosh[eB(s-t)]}{eB\sinh(eBs)}\right)\omega'^2$$

$$\left.+\left(\frac{(t-u)(s-t)}{s}-\frac{\sinh[eB(t-u)]\sinh[eB(s-t)]\cosh(eBu)}{eB\sinh(eBs)}\right)\omega''^2\right\}, \qquad (37)$$

with

$$D^\dagger[(\perp)\to(\|)'+(\|)'']=\bar{D}_1^\dagger(\omega,\omega',\omega'';s,t,u),$$
$$D^\dagger[(\|)\to(\perp)'+(\|)'']=\bar{D}_1^\dagger(-\omega',\omega'',-\omega;s,s-u,t-u),$$
$$D^\dagger[(\|)\to(\|)'+(\|)'']=\bar{D}_1^\dagger(-\omega'',-\omega,\omega';s,s-t+u,s-t),$$
$$D^\dagger[(\perp)\to(\perp)'+(\perp)'']=\bar{D}_2^\dagger(\omega,\omega',\omega'';s,t,u),$$
$$D^\dagger(\sigma\to\sigma'+\sigma'')=0 \qquad \text{otherwise}; \qquad (38)$$

where

$$\begin{aligned}
&\bar{D}_1^\dagger(\omega,\omega',\omega'';s,t,u)\\
&\quad=4m^2s^2\sinh^2(eBs)\{\omega'\sinh^2[eB(s-u)]+\omega''\sinh^2(eBt)\}\\
&\quad-4s\sinh^2(eBs)\{\omega'\sinh^2(eBu)+\omega''\sinh^2[eB(s-t)]\}\\
&\quad-8eBs^2\sinh(eBs)\{\omega'\sinh[eB(s-u)]\sinh(eBu)+\omega''\sinh(eBt)\\
&\quad\times\sinh[eB(s-t)\}+4\sinh^2(eBs)\{-\sinh^2[eB(s-u)][u\omega'-(s-t)\omega'']\\
&\quad\times(u\omega'+t\omega'')\omega'+\sinh^2(eBt)[u\omega'-(s-t)\omega''][(s-u)\omega'+(s-t)\omega'']\omega''\\
&\quad-\omega'\omega''s[\sinh^2(eBu)[(s-u)\omega'+(s-t)\omega'']+\sinh^2[eB(s-t)](u\omega'+t\omega'')]\}\\
&\quad+4s^2\{\omega'^3\sinh^2[eB(s-u)]\sinh^2(eBu)+\omega''^3\sinh^2[eB(s-t)]\sinh^2(eBt)\\
&\quad+\omega'^2\omega''\sinh[eB(s-t)]\sinh^2(eBu)\\
&\quad\times[2\sinh[eB(s-t)]+\sinh[eB(t+s-2u)]]\\
&\quad+\omega'\omega''^2\sinh(eBu)\sinh^2[eB(s-t)][2\sinh(eBu)+\sinh[eB(2t-u)]]\},\\
&\bar{D}_2^\dagger(\omega,\omega',\omega'';s,t,u)\\
&\quad=-4\sinh^2(eBs)\{m^2s^2+s-[u\omega'-(s-t)\omega''](u\omega'+t\omega'')\}\\
&\quad\times\{\sinh[eB(s-u)]\sinh[eB(2t-u-s)]\omega'\\
&\quad+\tfrac{1}{2}[\cosh[2eB(t-u)]-\cosh^2[eB(2t-u-s)]-\sinh^2[eB(2t-u-s)]]\omega''\}\\
&\quad+4\sinh^2(eBs)\{m^2s^2+s+(u\omega'+t\omega'')[(s-u)\omega'+(s-t)\omega'']\}\\
&\quad\times\{\tfrac{1}{2}[\cosh[2eB(s-t)]-\cosh^2[eB(s-t-u)]-\sinh^2[eB(s-t-u)]]\omega'\\
&\quad+\tfrac{1}{2}[\cosh(2eBu)-\cosh^2[eB(s-t-u)]-\sinh^2[eB(s-t-u)]]\omega''\}\\
&\quad-4\sinh^2(eBs)\{m^2s^2+s+[(s-u)\omega'+(s-t)\omega''][u\omega'-(s-t)\omega'']\}\\
&\quad\times\{\tfrac{1}{2}[\cosh[2eB(t-u)]-\cosh^2[eB(t-2u)]-\sinh^2[eB(t-2u)]]\omega'\\
&\quad+\sinh(eBt)\sinh[eB(t-2u)]\omega''\}
\end{aligned}$$

$$+4s^2\{\sinh^2(eBu)\sinh[eB(s-u)]\sinh[eB(2t-s-u)]\omega'^3$$
$$+\sinh^2[eB(s-t)]\sinh(eBt)\sinh[eB(t-2u)]\omega''^3$$
$$-[\sinh^2(eBu)\sinh^2[eB(s-t)]$$
$$+2\sinh(eBu)\sinh[eB(s-t)]\sinh(eBt)\sinh[eB(s-u)]]\omega\omega'\omega''\}. \qquad (39)$$

This form of the scattering amplitudes is valid below the threshold for pair production. The result for $M[(\perp)\to(\|)'+(\|)'']$ in the non-dispersive case below the threshold for pair production was obtained in a different form by Adler (1971) using the proper-time technique.

In the low-frequency limit the scattering amplitudes reduce to

$$M[(\perp)\to(\|)'+(\|)'']$$
$$=M[(\|)\to(\perp)'+(\|)'']=M[(\|)\to(\|)'+(\perp)'']$$
$$=-\frac{4\mathrm{i}\alpha^3B^3}{\pi^{1/2}}\frac{\omega\omega'\omega''}{m^8}\sin^3\theta M_1(B),$$
$$M[(\perp)\to(\perp)'+(\perp)'']=\frac{4\mathrm{i}\alpha^3B^3}{\pi^{1/2}}\frac{\omega\omega'\omega''}{m^8}\sin^3\theta M_2(B),$$
$$M(\sigma\to\sigma'+\sigma'')=0 \qquad \text{otherwise}; \qquad (40)$$

where

$$M_1(B):=\left(\frac{B_c}{B}\right)^4\int_0^\infty\frac{\mathrm{d}s}{s}\exp(-sB_c/B)\left\{\left(-\frac{3}{4s}+\frac{s}{6}\right)\coth s\right.$$
$$\left.+\left(\frac{1}{4}+\frac{s^2}{6}\right)\operatorname{cosech}^2 s+\frac{s}{2}\coth s\operatorname{cosech}^2 s\right\},$$
$$M_2(B):=\left(\frac{B_c}{B}\right)^4\int_0^\infty\frac{\mathrm{d}s}{s}\exp(-sB_c/B)\left\{\frac{3}{4s}\coth s\right.$$
$$\left.+\left(\frac{3}{4}-s^2\right)\operatorname{cosech}^2 s-\frac{3}{2}s^2\operatorname{cosech}^2 s\right\}. \qquad (41)$$

The corresponding absorption coefficients are

$$\kappa[(\perp)\to(\|)'+(\|)'']$$
$$=\kappa[(\|)\to(\perp)'+(\|)'']=\kappa[(\|)\to(\|)'+(\perp)'']$$
$$=\frac{\alpha^3m}{60\pi^2}\left(\frac{B\sin\theta}{B_c}\right)^6\left(\frac{\omega}{m}\right)^5(M_1(B))^2,$$
$$\kappa[(\perp)\to(\perp)'+(\perp)'']=\frac{\alpha^3m}{60\pi^2}\left(\frac{B\sin\theta}{B_c}\right)^6\left(\frac{\omega}{m}\right)^5(M_2(B))^2,$$
$$\kappa(\sigma\to\sigma'+\sigma'')=0 \qquad \text{otherwise}. \qquad (42)$$

These results for the splitting of a low-frequency photon were derived by Adler (1971) from the Heisenberg and Euler (1936) effective Lagrangian. The probability for photon splitting in the low-frequency limit is greatest when $\omega = \frac{1}{2}\omega' = \frac{1}{2}\omega''$, as in the weak-field limit. The scattering amplitudes and absorption coefficients in the low-frequency weak-field limit may be obtained from (40) and (42) by noting that

$$M_1(0) = \tfrac{26}{315}, \qquad M_2(0) = \tfrac{48}{315}. \tag{43}$$

The results obtained agree with those presented by Adler *et al* (1970) and by Bialynicka–Birula and Bialynicki–Birula (1970).

7. Dispersive effects

Conservation of 4-momentum for the photon splitting process $\sigma \to \sigma' + \sigma''$ requires that the refractive indices $\mu^{\sigma}(\omega, \theta)$ satisfy the 'index-matching' condition

$$(\omega' + \omega'')\mu^{\sigma}(\omega' + \omega'', \theta)\boldsymbol{K} = \omega'\mu^{\sigma'}(\omega', \theta')\boldsymbol{K}' + \omega''\mu^{\sigma''}(\omega'', \theta'')\boldsymbol{K}'', \tag{44}$$

where $\boldsymbol{K} := \boldsymbol{k}/|\boldsymbol{k}|$, $\theta = \cos^{-1}(\boldsymbol{K}\,.\,\boldsymbol{B}/B)$, etc. The photon splitting process is kinematically allowed when the triangle inequality for vector addition is satisfied by (44), that is, when

$$(\omega' + \omega'')\mu^{\sigma}(\omega' + \omega'', \theta) \leqslant \omega'\mu^{\sigma'}(\omega', \theta') + \omega''\mu^{\sigma''}(\omega'', \theta''). \tag{45}$$

The photon splitting process is kinematically forbidden when (45) is not satisfied.

In the weak-dispersion limit, Adler (1971) has shown analytically that the reactions $(\|) \to (\|)' + (\|)''$, $(\|) \to (\|)' + (\perp)''$, $(\|) \to (\perp)' + (\|)''$, $(\|) \to (\perp)' + (\perp)''$ and $(\perp) \to (\perp)' + (\perp)''$ are kinematically forbidden for $0 \leqslant \omega', \omega'' \leqslant \omega < 2m$ and $B < B_c$ and numerically that the reactions $(\perp) \to (\perp)' + (\|)''$, $(\perp) \to (\|)'' + (\perp)''$ and $(\perp) \to (\|)' + (\|)''$ are kinematically allowed for $0 \leqslant \omega', \omega'' \leqslant \omega \leqslant 1{\cdot}3\ m$ and $0 \leqslant B \leqslant 1{\cdot}3 B_c$. Adler concluded that, for $\omega < 2m$, the splitting of $\|$ polarised photons is absolutely forbidden by dispersive effects while the splitting of $\perp$ polarised photons is allowed. Adler also pointed out that the dominant splitting process for $\perp$ polarised photons in the weak-dispersion limit is $(\perp) \to (\|)' + (\|)''$ since the absorption coefficients for photon splitting processes involving an odd number of $\|$ polarised photons are very much smaller than those involving an even number of $\|$ polarised photons. It is shown in the Appendix that the contribution from the hexagon diagram dominates that from the box diagram for photon splitting in the weak-field weak-dispersion limit. The absorption coefficient for the process $(\perp) \to (\|)' + (\|)''$ is therefore approximately given by (42) with (43) in the low-frequency weak-field limit.

In general, the exact refractive indices must be used to determine whether a photon splitting process is kinematically allowed. The polarisation selection rules of Adler (1971) are inapplicable when dispersive effects are strong. Processes involving an odd number of | polarised photons and processes corresponding to the box diagram may give significant contributions to the exact scattering amplitudes for photon splitting.

Acknowledgments

This paper is based on part of my PhD thesis (Australian National University 1978). I would like to thank my supervisor Dr D B Melrose for many helpful discussions. The paper was written with the support of a CSIRO Postdoctoral Studentship.

Appendix

The weak-dispersion box diagram contribution to photon splitting

For weak photon dispersion (i.e. for $(k^2)_\perp \ll m^2$, $(k^2)_\parallel \ll m^2$, $(k'^2)_\perp \ll m^2$, etc) the box diagram contribution to the quadratic vacuum polarisation tensor reduces to

$$
\begin{aligned}
&\alpha^{\mu\nu\rho}(k, k', k'')\\
&\qquad = (e^4 B\mathrm{i}/1440\pi^2 m^4)[\beta^{\mu\nu\rho}(k, k', k'') + \beta^{\nu\mu\rho}(-k', -k, k'')\\
&\qquad\quad + \beta^{\rho\nu\mu}(-k'', k', -k)],
\end{aligned}
\tag{A.1}
$$

with

$$
\begin{aligned}
&\beta^{\mu\nu\rho}(k, k', k'')\\
&\qquad := k^2\{\hat{F}_\perp^{\nu\rho}(-9k_\parallel^\mu + 18k_\parallel'^\mu + k_\perp^\mu - 2k_\perp'^\mu)\\
&\qquad\quad + \hat{F}_\perp^{\rho\mu}(-3k_\parallel'^\nu - 9k_\perp^\nu + k_\perp'^\nu) + \hat{F}_\perp^{\mu\nu}(3k_\parallel^\rho - 3k_\parallel'^\rho + 8k_\perp^\rho + k_\perp'^\rho)\\
&\qquad\quad + 3g_\parallel^{\nu\rho}\hat{k}_\perp^\mu + g_\parallel^{\rho\mu}(12\hat{k}_\perp^\nu - 15\hat{k}_\perp'^\nu) + g_\parallel^{\mu\nu}(-3\hat{k}_\perp^\rho + 15\hat{k}_\perp'^\rho)\\
&\qquad\quad - 11g_\perp^{\nu\rho}\hat{k}_\perp^\mu + g_\perp^{\rho\mu}(7\hat{k}_\perp^\nu + 9\hat{k}_\perp'^\nu) + g_\perp^{\mu\nu}(16\hat{k}_\perp^\rho - 9\hat{k}_\perp'^\rho)\}\\
&\qquad\quad + (k''\hat{k}')_\perp[16g^{\nu\rho}(-k^\mu + 2k'^\mu) + 12g_\parallel^{\nu\rho}(-k_\parallel^\mu + 2k_\parallel'^\mu)]\\
&\qquad\quad - \hat{F}_\perp^{\nu\rho}\{k_\parallel^\mu[-6(k^2)_\parallel + 22(kk')_\parallel] + k_\parallel'^\mu[-10(k^2)_\parallel]\}\\
&\qquad\quad - g^{\nu\rho}\{\hat{k}_\perp^\mu[3(k^2)_\parallel + (kk')_\parallel - 9(k'^2)_\parallel + 6(k^2)_\perp - 6(kk')_\perp + 2(k'^2)_\perp]\\
&\qquad\quad + \hat{k}_\perp'^\mu[-8(k^2)_\parallel + 16(kk')_\parallel - 4(k^2)_\perp + 8(kk')_\perp]\}\\
&\qquad\quad - (g_\parallel^{\nu\rho} - g_\perp^{\nu\rho})\{\hat{k}_\perp^\mu[3(k^2)_\parallel + 5(kk')_\parallel - 13(k'^2)_\parallel] + \hat{k}_\perp'^\mu[-8(k^2)_\parallel + 16(kk')_\parallel]\}\\
&\qquad\quad + \hat{k}_\perp^\mu[12k_\parallel^\nu k_\parallel'^\rho - 40k_\parallel'^\nu k_\parallel'^\rho] + \hat{k}_\perp'^\mu[-28k_\parallel^\nu k_\parallel^\rho + 28k_\parallel^\nu k_\parallel'^\rho + 28k_\parallel'^\nu k_\parallel'^\rho]\\
&\qquad\quad + k_\parallel^\mu[-6k_\perp^\nu\hat{k}_\perp^\rho + 6\hat{k}_\perp^\nu k_\perp^\rho - 16k_\perp'^\nu\hat{k}_\perp^\rho - 10\hat{k}_\perp'^\nu k_\perp^\rho + 10k_\perp^\nu\hat{k}_\perp'^\rho]\\
&\qquad\quad + k_\parallel'^\mu[2k_\perp^\nu\hat{k}_\perp^\rho - 18\hat{k}_\perp^\nu k_\perp^\rho + 16k_\perp'^\nu\hat{k}_\perp^\rho + 16\hat{k}_\perp^\nu k_\perp'^\rho]\\
&\qquad\quad + \hat{k}_\perp^\mu[6k_\perp^\nu k_\perp^\rho - 10k_\perp'^\nu k_\perp^\rho - 4k_\perp^\nu k_\perp'^\rho + 6k_\perp'^\nu k_\perp'^\rho]\\
&\qquad\quad + \hat{k}_\perp'^\mu[-8k_\perp^\nu k_\perp^\rho + 8k_\perp'^\nu k_\perp^\rho + 8k_\perp^\nu k_\perp'^\rho].
\end{aligned}
\tag{A.2}
$$

The weak-dispersion scattering amplitudes corresponding to the box diagram may be derived from (A.1). For the kinematically allowed photon splitting processes one obtains

$$
\begin{aligned}
M[(\perp) \to (\parallel)' + (\parallel)''] &= \frac{\mathrm{i}\alpha e^3\omega}{45^2\pi^{3/2}m^2}\left(\frac{B\sin\theta}{B_\mathrm{c}}\right)^3[3\cos 2\beta(3\omega^2 - 14\omega'\omega'') - 14\omega'\omega''],\\
M[(\perp) \to (\perp)' + (\parallel)''] &= -\frac{\mathrm{i}\alpha e^3\omega''}{45^2\pi^{3/2}m^2}\left(\frac{B\sin\theta}{B_\mathrm{c}}\right)^3 3\sin 2\beta[3\omega^2 - 14\omega\omega' + 3\omega'^2],\\
M[(\perp) \to (\parallel)' + (\perp)''] &= -\frac{\mathrm{i}\alpha e^3\omega'}{45^2\pi^{3/2}m^2}\left(\frac{B\sin\theta}{B_\mathrm{c}}\right)^3 3\sin 2\beta[3\omega^2 - 14\omega\omega'' + 3\omega''^2],
\end{aligned}
\tag{A.3}
$$

where β is the dihedral angle between the plane containing the three photon directions and the plane containing the initial photon direction and the ambient field. The corresponding absorption coefficients are

$$\kappa[(\perp)\to(\|)'+(\|)'']=\frac{13\times 31\alpha^5 m}{60\times(90\pi)^4}\left(\frac{B\sin\theta}{B_c}\right)^6\left(\frac{\omega}{m}\right)^5,$$

$$\begin{aligned}\kappa[(\perp)\to(\perp)'+(\|)''] &= \kappa[(\perp)\to(\|)'+(\perp)''],\\ &= \frac{3\times 191\alpha^5 m}{140\times(90\pi)^4}\left(\frac{B\sin\theta}{B_c}\right)^6\left(\frac{\omega}{m}\right)^5.\end{aligned} \tag{A.4}$$

The absorption coefficients (A.4) are significantly smaller than the non-zero absorption coefficients given by the weak-field limit of (42). The contribution to photon splitting from the hexagon diagram dominates that from the box diagram in the weak-field weak-dispersion limit. The result (A.3) for $M[(\perp)\to(\|)'+(\|)'']$ agrees with that obtained from equation (A.2) of Bialynicka-Birula and Bialynicki-Birula (1970) and corrects that obtained by Adler (1971). The result (A.3) of Bialynicka-Birula and Bialynicki-Birula (1970) is obviously incorrect since it is not invariant under the transformation $\omega'\leftrightarrow\omega''$.

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